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NOTE

Weighted Inequalities on John Domains

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We study two-weighted inequalities on John domains. We first introduce domains that generalize the $\mathscr{C}(\lambda, M)$ domains defined by HajLasz and Koskela (*J. London Math. Soc.*, to appear) and then show that our domains are actually just John domains. We then extend an interesting and nice result of HajLasz and Koskela by modifying their proof and then obtain some interesting consequences that include weighted Sobolev interpolation inequalities and an almost necessary and sufficient condition for two-weighted Poincaré inequality on cubes.

Key Words: Poincaré inequalities; A_p weights; doubling; Sobolev interpolation inequalities; John domains; Boman chain condition.

1. INTRODUCTION

In 1982, Boman [2] introduced domains satisfying the Boman chain condition.

DEFINITION 1.1 [17]. An open set Ω in $\mathbb{R}^n$ is said to be a member of $\mathscr{F}(\sigma, N)$, $\sigma \geq 1$, $N \geq 1$, if there exists a covering W of Ω consisting of open cubes such that

(i) $$\sum_{Q \in W} \chi_{\sigma Q}(x) \leq N \chi_{\Omega}(x) \qquad \forall x \in \mathbb{R}^n.$$

(ii) There is a "central cube" $Q_0 \in W$ which can be connected with every cube $Q \in W$ by a finite chain of cubes $Q_0, Q_1, \ldots, Q_{k(Q)} = Q$ from

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W such that $Q \subset NQ_j$ for $j = 0, 1, \ldots, k(Q)$. Moreover, $Q_j \cap Q_{j+1}$ contains a cube R_j such that $Q_j \cup Q_{j+1} \subset NR_j$.

We say that Ω satisfies the Boman chain condition if $\Omega \in \mathscr{F}(\sigma, N)$ for some $N, \sigma \geq 1$. In 1985, Iwaniec and Nolder [17, Theorem 3] obtained an interesting inequality on this type of domain, which was then extended by Chua [7, Theorem 1.5] to doubling weight, who replaced the averages with polynomials.

It is easy to see that John domains satisfy the Boman chain condition. However, the converse is not obvious. It was until 1995 that Buckley et al. [4] showed that they are indeed equivalent. Moreover, in the same year, HajLasz and Koskela [14] introduced another chain condition and obtained an interesting result on domains satisfying the condition. The following is their chain condition. However, for simplicity, we restrict ourselves to Euclidean spaces.

DEFINITION 1.2. An open set $\Omega \in \mathscr{C}(\lambda, M)$, $\lambda \geq 1$, $M > 1$, if there exists a ball $B_0 \subset \Omega$ such that for all $x \in \Omega$, there is a sequence of balls $\{B_i\}$ such that

(i) $\lambda B_i \subset \Omega$ for all i and B_i is centered at x for all sufficiently large i;

(ii) $\operatorname{diam}(\Omega) M^{-1} 2^{-i} \leq r(B_i) \leq \operatorname{diam}(\Omega) M 2^{-i}$ for all i;

(iii) for each $i = 0, 1, 2, \ldots,$ there exists a ball $B_i' \subset B_i \cap B_{i+1}$ such that $B_i \cup B_{i+1} \subset MB_i'$.

Here and in what follows, B always denotes a ball, $r(B)$ is the radius of the ball, and tB is the ball concentric with B such that $r(tB) = tr(B)$. Moreover, we will denote by $B_r(x)$ the ball center at x with radius r.

The following is the interesting inequality obtained by HajLasz and Koskela [14, Theorem 1].

THEOREM 1.3. *Let $\Omega \in \mathscr{C}(\lambda, M)$. Assume that μ is a doubling measure. Assume that $g \in L^p_\mu(\Omega)$, $0 < p < \infty$, $u \in L^1_{loc,\mu}(\Omega)$ are such that the inequality*

$$\frac{1}{\mu(B)} \|u - u_{B,\mu}\|_{L^1_\mu(B)} \leq C_1 \frac{r(B)}{\mu(B)^{1/p}} \|g\|_{L^p_\mu(\lambda B)} \tag{1.1}$$

holds whenever $\lambda B \subset \Omega$. Then there exists $k > 1$ which depends only on p and the doubling constant C_μ of μ such that

$$\frac{1}{\mu(\Omega)^{1/kp}} \|u - u_{\Omega,\mu}\|_{L^{kp}_\mu(\Omega)} \leq C\big(C_1, C_\mu, p, k, \lambda, M\big) \frac{\operatorname{diam}(\Omega)}{\mu(\Omega)^{1/p}} \|g\|_{L^p_\mu(\Omega)}. \tag{1.2}$$

However, if $kp < 1$, $u_{\Omega,\mu}$ *has to be replaced by* $u_{B_{0,\mu}}$. *Note that* $u_{K,\mu} = \frac{1}{\mu(K)}\int_K u\,d\mu$.

It turns out that the ratio 2 in Definition 1.2(ii) is unnecessarily restrictive, even though cubes and balls are clearly $\mathscr{C}(\lambda, M)$ domain. An isosceles triangle with a sharp angle at the top is not a $\mathscr{C}(\lambda, M)$ domain. However, if we allow another ratio α, $\alpha > 1$ besides the ratio 2, then it will include many more domains. Indeed, now we have an equivalent definition of John domains (see Lemma 2.2).

DEFINITION 1.4. We say that an open set $\Omega \in \mathscr{C}(\lambda, M, \alpha)$, $\lambda \geq 1$, $\alpha, M > 1$, if there exists a "central ball" $B_0 \subset \Omega$ such that for all $x \in \Omega$, there is a sequence of balls $\{B_i\}$ such that

(i) $\lambda B_i \subset \Omega$ for all i and B_i is centered at x for all sufficiently large i;

(ii) $\operatorname{diam}(\Omega) M^{-1}\alpha^{-i} \leq r(B_i) \leq \operatorname{diam}(\Omega) M \alpha^{-i}$ for all i;

(iii) for each $i = 0, 1, 2, \ldots,$ there exists a ball $B_i' \subset B_{i+1}$ such that $B_i \cup B_{i+1} \subset MB_i'$.

Remark 1.5. Obviously, one can replace balls in the above definition with cubes, and radii of balls with edge lengths of cubes.

We will extend Theorem 1.3 to these domains with two weights and with the averages replaced by polynomials. Moreover, we are able to replace (1.1) with a much weaker assumption.

THEOREM 1.6. *Let* $\mathscr{D} \in \mathscr{C}(\lambda, M, \alpha)$, $\lambda \geq 1$, M, $\alpha > 1$, *and* $0 < p, s < \infty$. *Let u be a measurable function defined on* $\mathscr{D}$, *and let a be a nonnegative set function on all balls B with* $\lambda B \subset \mathscr{D}$. *Let* μ *be a doubling weight with doubling constant* C_μ. *Suppose there exists a doubling weight* σ *such that for any ball B with* $\lambda B \subset \mathscr{D}$, *there exists a polynomial* P_B *of degree* $\leq k$ *so that*

$$\frac{1}{\sigma(B)^{1/p_0}} \|u - P_B\|_{L^{p_0}_\sigma(B)} \leq a(B), \tag{1.3}$$

and there exist $0 < \delta < 1$ *such that*

$$\sum_{B \in W} a(B)^s \mu(B)^{1-\delta} \leq a_0^s\, \mu(\mathscr{D})^{1-\delta} \tag{1.4}$$

for any collection W of balls such that $\lambda B \subset \mathscr{D}$, $\lambda B_1 \cap \lambda B_2 = \emptyset$ *if* $B_1 \neq B_2$, $B_1, B_2 \in W$. *If there exists* $F \subset \mathscr{D}$ *such that* $\mu(\mathscr{D} \setminus F) = 0$ *and for all* $x \in F$,

$P_{B_r(x)}(x) \to u(x)$ *as* $r \to 0$, *then*

$$t^s \mu\left\{x \in \mathscr{D} : \left|u(x) - P_{B_0}(x)\right| > t\right\} \le C\left(M, \alpha, \sigma, C_\mu, p_0\right) a_0^s \, \mu(\mathscr{D}) \qquad \text{for all } t > 0, \quad (1.5)$$

where B_0 *is the "central ball" in* $\mathscr{D}$.

Remark 1.7. (i) Obviously, one could replace the balls B and their radii $r(B)$ with cubes Q and their edge lengths $l(Q)$, respectively.

(ii) Note that in the theorem, σ and μ are not related, except that they are both doubling weights.

(iii) It follows from (1.5) that we have the following corollary.

COROLLARY 1.8. *Under the assumption of Theorem* 1.6, *we have*

$$\frac{1}{\mu(\mathscr{D})^{1/q}} \|u - P_{B_0}\|_{L^q_\mu(\mathscr{D})} \le C\left(M, \alpha, \sigma, p_0, C_\mu, q\right) a_0$$

for all $0 < q < s$.

(iv) *Since* μ *is doubling*, *it is also reverse doubling*. *Hence*, *there exists* $k > 1$ *such that*

$$\left(\frac{\mu(B)}{\mu(\mathscr{D})}\right)^{1-1/k^2} \ge C\left(\frac{r(B)}{\operatorname{diam}(\mathscr{D})}\right)^p \qquad \text{for all balls } B \subset \mathscr{D}.$$

(*Note that* $\mathscr{D} \subset CB_0$.) *Thus*, *when* $a(B) = (r(B)/\mu(B)^{1/p})\|g\|_{L^p_\mu(\lambda B)}$, *where* g *is a measurable function on* $\mathscr{D}$, *we have that* (1.4) *holds with* $s = kp$, $\delta = 1 - \frac{1}{k}$, *and* $a_0 = C(\operatorname{diam}(\mathscr{D})/\mu(\mathscr{D})^{1/p})\|g\|_{L^p_\mu(\mathscr{D})}$. *Moreover*, *since* μ *is doubling*, *we know that* $(1/\mu(B_r(x)))\int_{B_r(x)} u \, d\mu \to u(x)$ *as* $r \to 0$ *for almost all* x. *Thus*, *Theorem* 1.3 (*by HajLasz and Koskela*) *is a special case of Corollary* 1.8.

By a weight w, we mean a nonnegative locally integrable function on $\mathbb{R}^n$. By abusing notation, we will also write w for the measure induced by w. Sometimes we write dw to denote $w\,dx$. We usually assume that w is doubling, by which we mean $w(2Q) \le Cw(Q)$ for every cube Q, where $2Q$ denotes the cube with the same center as Q and twice its edge length. All cubes in this paper are assumed to be closed and with edges parallel to the axes. By $w \in A_p$, we mean w satisfies the Muckenhoupt A_p condition, i.e.a, for all cubes Q,

$$\frac{1}{|Q|}\left(\int_Q w\,dx\right)^{1/p}\left(\int_Q w^{-1/(p-1)}\,dx\right)^{1/p'} \le C \qquad \text{when } 1 < p < \infty, \text{ and}$$

$$\frac{1}{|Q|}\int_Q w(x)\,dx \le C \operatorname*{essinf}_{x\in Q} w(x) \qquad \text{when } p = 1.$$

As usual, C denotes various positive constants, and $C(\alpha, \beta, \dots)$ denotes such constants depending only on $\alpha, \beta, \dots$. These constants may differ even in the same string of estimates.

2. PRELIMINARIES

First let us recall the definition of John domains.

DEFINITION 2.1. An open set $\mathscr{D}$ is a c-John domain if there exists a center $x_0 \in \mathscr{D}$ such that for all $x \neq x_0$, $x \in \mathscr{D}$, there exists a rectifiable curve γ: $[0, l] \to \mathscr{D}$ parameterized by arc length such that

$$\operatorname{dist}(\gamma(t), \partial\mathscr{D}) > ct, \qquad \text{for all } 0 \le t \le l.$$

LEMMA 2.2. *$\mathscr{D}$ is a John domain if and only if there exists $\alpha, M, \lambda > 1$ such that $\mathscr{D} \in \mathscr{C}(\lambda, M, \alpha)$.*

Proof. First it is easy to show that if $\mathscr{D} \in \mathscr{C}(\lambda, M, \alpha)$, then $\mathscr{D}$ is a John domain (see the proof of [4, Theorem 3.1]). Next, suppose $\mathscr{D}$ is a c-John domain. We may assume $0 < c < 1$. Let x_0 be the "center" of $\mathscr{D}$, and

$$d = \operatorname{diam}(\mathscr{D}), \qquad L = \sup_{x\in\mathscr{D}} |x - x_0|.$$

Note that $L \le d \le 2L$. For any $x \in \mathscr{D}$, $x \neq x_0$, let γ: $[0, l] \to \mathscr{D}$ be a curve parameterized by its arc length such that $\gamma(0) = x$, $\gamma(l) = x_0$, and

$$\operatorname{dist}(\gamma(t), \partial\mathscr{D}) > ct \qquad \text{for all } 0 \le t \le l.$$

It is clear that $l < L/c$. Next, there exists $r > 0$ such that $B_{r/c}(x) \subset \mathscr{D}$. Let $t' \in (0, l)$ such that $\gamma(t') \in B_r(x)$ and $r \le t' < r/c^2$. Choose nonnegative integers k, k' such that

$$(1 + c^3)^k < l/t' \le (1 + c^3)^{k+1}, \text{ and } (1 + c^3)^{k'} < L/cl \le (1 + c^3)^{k'+1}.$$

Let $t_k = t'$, $t_{k-1} = t_k(1 + c^3), \dots, t_1 = t_2(1 + c^3)$, and $t_0 = l$. We will now define the chain of balls for x. First, let $B_0 = B_{c^2L/2}(x_0)$ and $B_i = (1 + c^3)^{-i}B_0$ for $i = 1, 2, \dots, k' - 1$. Moreover, let $B_{k'} = B_{c^3 l}(x_0)$ and $B_{i+k'} = B_{c^3 t_i}(\gamma(t_i))$ for $i = 1, 2, \dots, k$. Finally, let $B_{k+k'+i} = (1 + c^3)^{1-i}B_r(x)$ for $i = 1, 2, \dots$.

It is clear that there exists $M_1 > 1$ such that

$$dM_1^{-1}(1 + c^3)^{-i} \le r(B_i) \le dM_1(1 + c^3)^{-i}$$

and $\frac{1}{c}B_i \subset \mathscr{D}$ for all i. Moreover, note that for $i = 0, 1, \dots, k - 1$, $\gamma(t_i) \in B_{k'+i+1}$ as $|\gamma(t_i) - \gamma(t_{i+1})| \le t_i - t_{i+1} \le c^3 t_{i+1}$. Hence, it is clear that the center of B_i is in B_{i+1} for all i. Thus, there exists $M_2 > 1$ so that for each i, there exists a ball $B_i' \subset B_i \cap B_{i+1}$ such that $B_i \cup B_{i+1} \subset M_2(M_1)^n B_i'$. Hence, $\mathscr{D} \in \mathscr{C}(1/c, M, 1 + c^3)$ for some $M > 1$.

Next, let us state some inequalities on polynomials.

LEMMA 2.3 [23, Chap. 3, Lemma 7]. *If w is a doubling measure and m is a positive integer, then there exists $s_0(n, m, w)$ such that if $s < s_0$ then for all cubes Q, $\lambda > 0$ such that*

$$w(\{x \in Q : |p(x)| > \lambda\}) \le sw(Q)$$

we have

$$\sup_{x \in Q} |p(x)| \le C\lambda,$$

where p is any polynomial of degree $\le m$ and C is a constant independent of λ, Q, and p.

It follows from Chebyshev's inequality and this lemma that given m and a polynomial p of degree $\le m$.

$$\|p\|_{L^\infty(Q)} \le \frac{C}{w(Q)^{1/q}} \|p\|_{L^q_w(Q)}, \qquad 0 < q < \infty,$$

with C independent of Q and p.

LEMMA 2.4 [6, Theorem 2.2]. *Let Q be a cube and let E be a measurable set in Q with $|E| > \gamma|Q|$. If p is a polynomial of degree $\le m$, then*

$$\|p\|_{L^\infty(E)} \ge C(\gamma, m)\|p\|_{L^\infty(Q)}.$$

Remark 2.5. Clearly, one could replace all of the cubes in the above two lemmas with balls.

3. PROOF OF MAIN RESULTS

Proof of Theorem 1.6. We will modify the proof in [14]. Let $A_t = \{x \in \mathscr{D} : |u(x) - P_{B_0}(x)| > t\}$, $t > 0$. If $x \in F \cap A_t$, let $B_1, B_2, \ldots$ be the chain associated to x. Let x_i be the center of B_i and $r(B_i) = r_i$. Note that for all i, we have

$$|x - x_i| \le \sum_{j=i}^{\infty} |x_j - x_{j+1}| \le 2 \sum_{j=i}^{\infty} r_j < \frac{2M^2 r_i}{1 - \alpha^{-1}}.$$

Also recall that $P_{B_i}(x) \to u(x)$ as $i \to \infty$. Hence, by the triangle inequality,

$$
\begin{aligned}
t &\le \left| u(x) - P_{B_0}(x) \right| \le \sum_{i=0}^{\infty} \left| P_{B_i}(x) - P_{B_{i+1}}(x) \right| \\
&\le \sum_{i=0}^{\infty} \left(\left| P_{B_i}(x) - P_{B_i'}(x) \right| + \left| P_{B_i'}(x) - P_{B_{i+1}}(x) \right| \right) \\
&\le \sum_{i=0}^{\infty} \left(\|P_{B_i} - P_{B_i'}\|_{L^\infty(KB_i')} + \|P_{B_{i+1}} - P_{B_i'}\|_{L^\infty(KB_i')} \right) \\
&\qquad\qquad \text{since } x \in \frac{2M^2}{1 - \alpha^{-1}} B_i \subset \frac{2M^3}{1 - \alpha^{-1}} B_i' = KB_i' \text{ for all } i \\
&\le C(M, \alpha) \sum_{i=0}^{\infty} \left(\|P_{B_i} - P_{B_i'}\|_{L^\infty(B_i')} + \|P_{B_{i+1}} - P_{B_i'}\|_{L^\infty(B_i')} \right) \qquad \text{by Lemma 2.4} \\
&\le C(M, \alpha, \sigma, p_0, k) \sum_{i=0}^{\infty} \frac{1}{\sigma(B_i')^{1/p_0}} \\
&\qquad \times \left(\|P_{B_i} - P_{B_i'}\|_{L^{p_0}_\sigma(B_i')} + \|P_{B_{i+1}} - P_{B_i'}\|_{L^{p_0}_\sigma(B_i')} \right) \qquad \text{by Lemma 2.3} \\
&\le C(M, \alpha, \sigma, p_0, k) \sum_{i=0}^{\infty} \frac{2}{\sigma(B_i')^{1/p_0}} \\
&\qquad \times \left(\|P_{B_i} - u\|_{L^{p_0}_\sigma(B_i')} + 2\|P_{B_i'} - u\|_{L^{p_0}_\sigma(B_i')} + \|P_{B_{i+1}} - u\|_{L^{p_0}_\sigma(B_i')} \right) \\
&\le C(M, \alpha, \sigma, p_0, k) \sum_{i=0}^{\infty} \frac{1}{\sigma(B_i')^{1/p_0}} \\
&\qquad \times \left(\|P_{B_i} - u\|_{L^{p_0}_\sigma(B_i)} + 2\|P_{B_i'} - u\|_{L^{p_0}_\sigma(B_i')} + \|P_{B_{i+1}} - u\|_{L^{p_0}_\sigma(B_{i+1})} \right) \\
&\le C(M, \alpha, \sigma, p_0, k) \sum_{i=0}^{\infty} \left[a(B_i) + a(B_i') + a(B_{i+1}) \right]
\end{aligned}
$$

by (1.3). Next, since μ is doubling, there exists $\varepsilon > 0$ such that

$$\left(\frac{\mu(B)}{\mu(\mathscr{D})}\right)^{\delta/s} \leq C\left(\frac{r(B)}{d}\right)^{\varepsilon} \qquad \text{for all balls } B \subset \mathscr{D},$$

whee $d = \operatorname{diam}(\mathscr{D})$. Moreover,

$$M^{-\varepsilon} \sum_{i=0}^{\infty} t\alpha^{-i\varepsilon} \leq \sum_{i=0}^{\infty} \left[a(B_i) + a(B_i') + a(B_{i+1})\right].$$

Thus, there exists i such that

$$t(r_i/d)^{\varepsilon} \leq C\left[a(B_i) + a(B_i') + a(B_{i+1})\right],$$

and hence, $(\mu(B_i)/\mu(\mathscr{D}))^{\delta} \leq Ct^{-s}[a(B_i) + a(B_i') + a(B_{i+1})]^s$. Therefore,

$$\mu(B_i) \leq Ct^{-s}\left[a(B_i) + a(B_i') + a(B_{i+1})\right]^s \mu(B_i)^{1-\delta}\mu(\mathscr{D})^{\delta}.$$

Now since $|x - x_i| < 2M^2 r_i/(1 - \alpha^{-1})$, if $B_x = B_{r_i}(x)$, then $B_x \subset (2M^2/(1 - \alpha^{-1}) + 1)B_i$. Thus,

$$\begin{aligned}
\mu(B_x) &\leq C(M, C_\mu)\mu(B_i) \\
&\leq C(M, \alpha, C_\mu, \sigma, p_0)t^{-s} \\
&\qquad \times \left[a(B_i) + a(B_i') + a(B_{i+1})\right]^s \mu(\mathscr{D})^{\delta}\mu(B_i)^{1-\delta} \\
&\leq Ct^{-s}\Big[a(B_i)^s \mu(B_i)^{1-\delta} + a(B_i')^s \mu(B_i')^{1-\delta} \\
&\qquad\qquad + \mu(B_{i+1})^s \mu(B_{i+1})^{1-\delta}\Big]\mu(\mathscr{D})^{\delta},
\end{aligned}$$

since μ is doubling. Next, clearly, collection of B_x is a covering of $A_t \cap F$. Hence, by the Vitali type covering lemma (again since μ is doubling), there exist disjoint balls $B_{y_1}, B_{y_2}, \ldots, B_{y_n}$ such that $\mu(A_t) = \mu(A_t \cap F) \leq C\sum_{j=1}^{n} \mu(B_{y_j})$. It then follows from (1.4) that

$$\mu(A_t) \leq C(M, \alpha, C_\mu, \sigma, p_0, k)t^{-s}a_0^s\,\mu(\mathscr{D}).$$

4. APPLICATIONS

In particular, when $\mathscr{D}$ is a cube (or ball), we have some interesting consequences.

THEOREM 4.1. *Let u be a measurable function and let v be a nonnegative measure. Suppose there exist $\varepsilon > 0$, $0 < p_0 < \infty$, and a doubling weight σ such that*

$$\frac{1}{\sigma(Q)^{1/p_0}}\|u - u_{Q,\sigma}\|_{L^{p_0}_{\sigma}(Q)} \leq v(Q)^{\varepsilon}$$

for all cubes Q. Then, for any doubling weight μ and any $0 < q < \infty$, we have for all cubes Q,

$$\frac{1}{\mu(Q)^{1/q}}\|u - u_{Q,\mu}\|_{L^q_\mu(Q)} \le Cv(Q)^{\varepsilon}.$$

Proof. We will apply Corollary 1.8 with balls replaced by cubes. It suffices to consider the case $q \ge 1$. Let μ be any doubling weight. We now choose $s > q$ such that $s\varepsilon \ge 1$. Let $a(Q) = v(Q)^{\varepsilon}$. It is now easy to see that

$$\sum_{Q \in W} v(Q)^{s\varepsilon}\mu(Q)^{1/2} \le v(\tilde{Q})^{s\varepsilon}\mu(\tilde{Q})^{1/2}$$

for any collection W of disjoint subcubes of $\tilde{Q}$. Hence, by Corollary 1.8, let Q' be the "central cube" in Q. We have

$$\frac{1}{\mu(Q)^{1/p}}\|u - u_{Q',\sigma}\|_{L^q_\mu(Q)} \le Cv(Q)^{\varepsilon}.$$

Finally, note that since $q \ge 1$, by the triangle inequality and Hölder's inequality, we have

$$\frac{1}{\mu(Q)^{1/q}}\|u - u_{Q,\mu}\|_{L^q_\mu(Q)} \le \frac{2}{\mu(Q)^{1/q}}\|u - u_{Q',\sigma}\|_{L^q_\mu(Q)},$$

and this proves the theorem.

Remark 4.2. Clearly, the averages can be replaced by suitable polynomials in the above theorem, and of course the theorem remains valid if we replace cubes with balls.

THEOREM 4.3. *Let $0 < p < q_0 < \infty$. Let μ be a doubling weight and let v and w be weights such that*

$$\left(\frac{l(Q)}{l(\tilde{Q})}\right)\left(\frac{\mu(Q)}{\mu(\tilde{Q})}\right)^{1/q_0} \le C\left(\frac{w(Q)}{w(\tilde{Q})}\right)^{1/p} \quad \text{for all cubes } Q, \tilde{Q} \text{ such that } Q \subset \tilde{Q}, \tag{4.1}$$

and there exist a doubling weight σ and $p_0 > 0$ such that for all cubes Q,

$$\frac{1}{\sigma(Q)^{1/p_0}}\|u - u_{Q,v}\|_{L^{p_0}_\sigma(Q)} \le C\frac{l(Q)}{w(Q)^{1/p}}\|g\|_{L^p_w(Q)}, \tag{4.2}$$

where u is a Lipschitz continuous function and g is a measurable function. Then, for all $0 < q < q_0$,

$$\frac{1}{\mu(Q)^{1/q}}\|u - u_{Q,v}\|_{L^q_\mu(Q)} \leq C\frac{l(Q)}{w(Q)^{1/p}}\|g\|_{L^p_w(Q)} \tag{4.3}$$

for all cubes Q. Hence, if $1 \leq r \leq p$, $g = |\nabla u|$ and, in addition, v and w are doubling, then there exists $h > 1$ such that

$$\frac{1}{\mu(Q)^{1/ph}}\|u\|_{L^{ph}_\mu(Q)} \leq C\left(\frac{\|u\|_{L^r_v(Q)}}{v(Q)^{1/r}}\right)^{(h-1)/h}\left(\frac{\|u\|_{L^r_v(Q)}}{v(Q)^{1/r}} + \frac{l(Q)\|\nabla u\|_{L^p_w(Q)}}{w(Q)^{1/p}}\right)^{1/h}. \tag{4.4}$$

Remark 4.4. (i) It is well known that when $g = |\nabla u|$, $1 \leq p \leq q < \infty$, and w is doubling, if (4.3) holds for all Lipschitz continuous functions u, then

$$\frac{l(Q)}{l(\tilde{Q})}\left(\frac{\mu(Q)}{\mu(\tilde{Q})}\right)^{1/q} \leq C\left(\frac{w(Q)}{w(\tilde{Q})}\right)^{1/p} \tag{4.5}$$

for all cubes $Q \subset \tilde{Q}$ (see for example [5, p. 1192]; however, the condition $1 \leq p \leq q < \infty$ is indeed not needed). Hence, the condition (4.1) is almost necessary.

(ii) Again, one can replace cubes and edge lengths of cubes with balls and radii of balls, respectively, in Theorem 4.2.

(iii) Suppose v, μ, and w are doubling weights, $1 \leq p < \infty$, and $g = |\nabla u|$. In [9, Corollary 1.6], we know that (4.4) holds if there exists $q > p$ such that (4.3) holds. However, (4.3) actually implies (4.2) and (4.5). Thus, Theorem 4.2 is an extension of Corollary 1.6 in [9] and, hence, an extension of Theorem 1 of [13]. For further extension, see Theorem 4.5.

Proof. We need only prove the first part, as the second part is just a consequence of the first part by [9, Corollary 1.6]. We will again apply Corollary 1.8 with balls replaced by cubes. First, clearly, any cube is a $\mathscr{C}(1, M, \alpha)$ domain. Now, let $a(Q) = (l(Q)/(w(Q)^{1/p}))\|g\|_{L^p_w(Q)}$. Fix any cube Q_0; then (1.4) holds with $s = q_1$ and $\delta = q_1/q_0$ for any $p \leq q_1 < q_0$, with $a_0 = (l(Q_0)/(w(Q_0)^{1/p}))\|g\|_{L^p_w(Q_0)}$. Hence, by Corollary 1.8, for any $0 < q < q_0$, we have

$$\frac{1}{\mu(Q_0)^{1/q}}\|u - u_{Q_0',v}\|_{L^q_\mu(Q_0)} \leq C\frac{l(Q_0)}{w(Q_0)^{1/p}}\|g\|_{L^p_w(Q_0)},$$

where Q_0' is the "central cube" of Q_0 and, hence, $Q_0 \subset CQ_0'$. Finally, note that

$$
\begin{aligned}
&\frac{1}{\mu(Q_0)^{1/q}}\|u-u_{Q_0,v}\|_{L^q_\mu(Q_0)}\\
&\quad\le \frac{C}{\mu(Q_0)^{1/q}}\|u-u_{Q_0',v}\|_{L^q_\mu(Q_0)}+\frac{C}{\mu(Q_0)^{1/q}}\|u_{Q_0,v}-u_{Q_0',v}\|_{L^q_\mu(Q_0)}\\
&\quad= \frac{C}{\mu(Q_0)^{1/q}}\|u-u_{Q_0',v}\|_{L^q_\mu(Q_0)}+\frac{C}{\sigma(Q_0')^{1/p_0}}\|u_{Q_0,v}-u_{Q_0',v}\|_{L^{p_0}_\sigma}(Q_0')\\
&\quad\le \frac{C}{\mu(Q_0)^{1/q}}\|u-u_{Q_0',v}\|_{L^q_\mu(Q_0)}+\frac{C}{\sigma(Q_0')^{1/p_0}}\|u-u_{Q_0',v}\|_{L^{p_0}_\sigma(Q_0')}\\
&\qquad+\frac{C}{\sigma(Q_0')^{1/p_0}}\|u-u_{Q_0,v}\|_{L^{p_0}_\sigma(Q_0')}\\
&\quad\le \frac{C}{\mu(Q_0)^{1/q}}\|u-u_{Q_0',v}\|_{L^q_\mu(Q_0)}+\frac{C}{\sigma(Q_0')^{1/p_0}}\|u-u_{Q_0',v}\|_{L^{p_0}_\sigma(Q_0')}\\
&\qquad+\frac{C}{\sigma(Q_0)^{1/p_0}}\|u-u_{Q_0,v}\|_{L^{p_0}_\sigma(Q_0)}
\end{aligned}
$$

since σ is doubling. The theorem is now clear.

Next, by a similar approach, in the case $g=|\nabla u|$, we have

THEOREM 4.5. *Let* $\le p<q_0<\infty$. *Let* μ *be a doubling weight such that*

$$
\left(\frac{l(Q)}{l(\tilde Q)}\right)^{1-n}\left(\frac{\mu(Q)}{\mu(\tilde Q)}\right)^{1/q_0}\left(\frac{w^{-1/(p-1)}(Q)}{w^{-1/(p-1)}(\tilde Q)}\right)^{1/p'}\le C
$$

for all cubes $Q, \tilde Q$ *in* $\mathbb{R}^n$ *such that* $Q\subset\tilde Q$.

Then, for all $0<q<q_0$ *and all cubes* Q,

$$
\frac{1}{\mu(Q)^{1/q}}\|u-u_Q\|_{L^q_\mu(Q)}\le Cl(Q)^{1-n}\left(w^{-1/(p-1)}(Q)\right)^{1/p'}\|\nabla u\|_{L^p_w(Q)} \tag{4.7}
$$

for all Lipschitz continuous functions u. *Here*

$$\big(w^{-1/(p-1)}(Q)\big)^{1/p'} = \operatorname*{esssup}_{x\in Q} w(x)^{-1} \qquad \textit{when } p = 1 \textit{ and}$$

$$\big(w^{-1/(p-1)}(Q)\big)^{1/p'} = \left(\int_Q w(x)^{-1/(p-1)}\,dx\right)^{1/p'} \qquad \textit{when } p > 1.$$

Proof. First, note that by the nonweighted Poincaré inequality and Hölder's inequality,

$$\begin{aligned}\frac{1}{|Q|}\|u - u_Q\|_{L^1(Q)} &\le C\frac{l(Q)}{|Q|}\|\nabla u\|_{L^1(Q)}\\ &\le Cl(Q)^{1-n}\big(w^{-1/(p-1)}(Q)\big)^{1/p'}\|\nabla u\|_{L^p_w(Q)}.\end{aligned}$$

We can now proceed as in the proof of the previous theorem to show that (4.7) holds.

Remark 4.6. Under the assumption of Theorem 4.5, if, in addition, $w^{-1/(p-1)}$ is reverse doubling, then it follows from [21, Theorem 1B] that (4.7) holds even for $q = q_0$.

THEOREM 4.7. *Let* $0 < p_1$, $p_2 \le q < \infty$, λ, $p_2 \ge 1$, *and* $0 < \alpha < \beta$. *Suppose that* μ, v *are doubling weights and* w *is a weight. Let* u *be a Lipschitz continuous function and let* g *be a measurable function. Suppose further that*

$$\left(\frac{l(Q)}{l(\tilde{Q})}\right)^{\alpha}\left(\frac{\mu(Q)}{\mu(\tilde{Q})}\right)^{1/q}\left[\left(\frac{w(Q)}{w(\tilde{Q})}\right)^{-1/p_1} + \left(\frac{v(Q)}{v(\tilde{Q})}\right)^{-1/p_2}\right] \le C \tag{4.8}$$

for all cubes $Q \subset \tilde{Q}$. *If there exist a doubling weight* σ *and* $0 < p_0$ *such that*

$$\frac{1}{\sigma(Q)^{1/p_0}}\|u - u_{Q,v}\|_{L^{p_0}_\sigma(Q)} \le C\frac{l(Q)^{\beta}}{w(Q)^{1/p_1}}\|g\|_{L^{p_1}_w(\lambda Q)} \tag{4.9}$$

for all cubes Q, *then for all cubes* Q, *we have*

$$\begin{aligned}&\frac{1}{\mu(Q)^{1/q}}\|u\|_{L^q_\mu(Q)}\\ &\quad\le C\left(\frac{\|u\|_{L^{p_2}_v(Q)}}{v(Q)^{1/p_2}}\right)^{(\beta-\alpha)/\beta}\left(\frac{\|u\|_{L^{p_2}_v(Q)}}{v(Q)^{1/p_2}} + \frac{l(Q)^{\beta}\|g\|_{L^{p_1}_w(Q)}}{w(Q)^{1/p_1}}\right)^{\alpha/\beta}.\end{aligned}$$

Proof. Let Q_0 be a cube. First, note that $Q_0 \in \mathscr{C}(\lambda, M, \alpha)$ for some M, $\alpha > 1$. Since μ is doubling, by (4.6), there exists $s > q$ such that

$$\left(\frac{l(Q)}{l(\tilde{Q})}\right)^{\beta}\left(\frac{\mu(Q)}{\mu(\tilde{Q})}\right)^{1/2} \le C\left(\frac{w(Q)}{w(\tilde{Q})}\right)^{1/p_1}.$$

Again, by Corollary 1.8, we have if Q_0' is the "central cube" in Q_0, then

$$\frac{1}{\mu(Q_0)^{1/q}}\|u - u_{Q_0',v}\|_{L^q_\mu(Q_0)} \le C\frac{l(Q_0)^\beta}{w(Q_0)^{1/p_1}}\|g\|_{L^{p_1}_w(Q_0)}.$$

Hence, if $Q_0 \subset Q$, by the triangle inequality and Hölder's inequality,

$$\begin{aligned}
\|u\|_{L^q_\mu(Q_0)} &\le C\mu(Q_0)^{1/q}v(Q_0')^{-1/p_2}\|u\|_{L^{p_2}_v(Q_0')} \\
&\qquad + C\mu(Q_0)^{1/q}l(Q_0)^\beta w(Q_0)^{-1/p_1}\|g\|_{L^{p_1}_w(Q_0)} \\
&\le C\mu(Q_0)^{1/q}v(Q_0)^{-1/p_2}\|u\|_{L^{p_2}_v(Q_0)} \\
&\qquad + C\mu(Q_0)^{1/q}l(Q_0)^\beta w(Q_0)^{-1/p_1}\|g\|_{L^{p_1}_w(Q_0)}
\end{aligned}$$

since v is doubling. Next (using the same argument as in [9]), we may decompose any cube Q into unions of nonoverlapping subcubes $\{Q_0\}$ such that $l(Q_0)/l(Q) = \gamma$. Then by (4.9),

$$\begin{aligned}
\|u\|^q_{L^q_\mu(Q)} &= \sum_{Q_0}\|u\|^q_{L^q_\mu(Q_0)} \\
&\le C\left(\mu(Q)^{1/q}v(Q)^{-1/p_2}\right)^q\gamma_{Q_0}^{-\alpha q}\sum_{Q_0}\|u\|^q_{L^{p_2}_v(Q_0)} \\
&\qquad + C\left(\mu(Q)^{1/q}l(Q)^\beta w(Q)^{-1/p_1}\right)^q\gamma^{(\beta-\alpha)q}\sum_{Q_0}\|g\|^q_{L^{p_1}_w(Q_0)} \\
&\le C\left(\mu(Q)^{1/q}v(Q)^{-1/p_2}\right)^q\gamma^{-\alpha q}\left(\sum_{Q_0}\|u\|^{p_2}_{L^{p_2}_v(Q_0)}\right)^{q/p_2} \\
&\qquad + C\left(\mu(Q)^{1/q}l(Q)^\beta w(Q)^{1/p_1}\right)^q\gamma^{(\beta-\alpha)q}\left(\sum_{Q_0}\|g\|^{p_1}_{L^{p_1}_w(Q_0)}\right)^{q/p_1}
\end{aligned}$$

since $q \ge p_1,\ p_2$. Hence,

$$\begin{aligned}
\frac{1}{\mu(Q)^{1/q}}\|u\|_{L^q_\mu(Q)} &\le C\gamma^{-\alpha}v(Q)^{-1/p_2}\|u\|_{L^{p_2}_v(Q)} \\
&\qquad + C\gamma^{\beta-\alpha}w(Q)^{-1/p_1}l(Q)^\beta\|g\|_{L^{p_1}_w(Q)}.
\end{aligned}$$

It is now easy to see that the theorem holds by choosing an appropriate γ. For more details, refer to [9].

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